Wake field generation and nonlinear evolution in a magnetized electron-positron-ion plasma

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The nonlinear propagation of a circularly polarized electromagnetic (CPEM) wave in a strongly magnetized electron-positron-ion plasma is investigated. Two coupled equations describing the interaction between a high-frequency CPEM wave and the low-frequency electrostatic wake field are derived. It is found that the generation of the wake fields depends on the presence of the ion species. The wake field generation in turn leads to de-acceleration and frequency down conversion of the electromagnetic pulse.

I. INTRODUCTION

Electron-positron (pair) plasmas are composed of electrons and positrons which have the same mass and opposite charge. Such pair plasmas appeared in the early Universe [1, 2], and are frequently encountered in bipolar outflows (jets) in active-galactic nuclei [3, 4], in micro-quasars [5], in pulsar magnetospheres [6, 7, 8, 9], in magnetars [10], in cosmological gamma ray fireballs [11], in solar flares [12], and at the center of our galaxy [13]. Multiterawatt and petawatt laser pulses interacting with solid density matters can create pair plasmas [14, 15, 16, 17]. Colliding electromagnetic pulses also generate pairs from vacuum [18]. The pair plasmas at the surface of fast rotating neutron stars and magnetars are held in strong magnetic fields, while super-strong magnetic fields can be created in intense laser-plasma interaction experiments. Accordingly, the understanding of collective phenomena in strongly magnetized pair plasmas has been a topic of significant interest [19, 20, 21, 22]. Specifically, it is to be noted that in a magnetized pair plasma, we have new wave modes whose counterparts do not exist in an electron-ion magnetoplasma [23]. Furthermore, both astrophysical and laboratory plasmas contain a fraction of ions besides the electrons and positrons. In such environments, the linear and nonlinear wave propagation characteristics are modified [24, 25, 26, 27, 28, 29, 30, 31, 32].

In this paper, we consider the nonlinear propagation of magnetic field-aligned circularly polarized electromagnetic (CPEM) waves in a pair-ion magnetoplasma, to show the possibility of exciting electrostatic plasma wakes by the ponderomotive force of the CPEM waves. Our manuscript is organized in the following fashion. In Sec. II we discuss the linear dispersion relation, the group velocity and the group dispersion for the magnetic field-aligned CPEM waves in a pair-ion magnetoplasma. Section III considers the nonlinear interactions between finite amplitude CPEM waves and electrostatic plasma oscillations (EPOs), and presents the coupled equations. Nonlinearities associated with the electron and positron mass variations, as well as with the CPEM ponderomotive force are incorporated in the dynamical equations for the modulated CPEM waves and the driven EPOs. Section IV presents the conservation laws for nonlinearly coupled CPEM wave EPO systems, and discusses the excitation of wakefields . Section V contains a brief summary of our investigation.

II. THE LINEAR DISPERSION RELATION

We consider a magnetized quasi-neutral electron-positron-ion plasma, i.e. $n_{0e} = n_{0i} + n_{0p}$, where n_{0e} is the background electron number density, n_{0p} is the background positron density, n_{0i} is the background ion density, and the background magnetic field is given by $\vec{B}_0 = B_0\hat{z}$. For a right hand circularly polarized electromagnetic (RCPEM) wave propagating along $\hat{z}$ with frequency $\omega \gg \omega_{pi} = (4\pi n_{0i}e^2/m_i)^{1/2}$, we have the dispersion relation [33, 34]

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega(\omega+\omega_{ce})} - \frac{\omega_{pp}^2}{\omega(\omega+\omega_{cp})}, \tag{1}$$

where $\omega_{pj} = (4\pi n_{0j}e^2/m_e)^{1/2}$, with $j = e, p$, is the plasma frequency, $\omega_{ce} = -eB_0/m_e = -\omega_{cp}$ is the electron gyrofrequency, e is the magnitude of the electron charge, and m_e is the electron mass (for left hand polarization $\omega + \omega_c$ is replaced by $\omega - \omega_c$ [35]). In the limit of a strong background magnetic field, i.e. $\omega \ll |\omega_{ce}|$, we can simplify (1) to obtain

$$\frac{k^2c^2}{\omega^2} = 1 + \frac{\Omega_i^2}{\omega\omega_{cp}}, \tag{2}$$

where we have introduced the notation $\Omega_i = (4\pi n_{0i}e^2/m_e)^{1/2}$ and used the background charge neutrality condition. From Eq. (2) we obtain the group velocity

$$v_g = \frac{kc^2}{\omega + \Omega_i^2/2\omega_{cp}}, \tag{3}$$

and the group velocity dispersion $v_g' = dv_g/dk = (v_g/k)(1-v_g^2/c^2)$. In the limit $kc \gg \omega_p^2/\omega_{cp}$, we obtain

$$v_g \approx c\left[1 - \frac{1}{2}\left(\frac{\Omega_i^2}{2\omega_{cp}kc}\right)^2\right], \tag{4}$$

and $v_g' \approx \Omega_i^2/4\omega_{cp}^2k^3c$, and we have neglected the root with negative sign of the group velocity. Thus, in this limit the RCPEM waves propagate with a speed close to the speed of light.

III. NONLINEAR EVOLUTION EQUATIONS

Next, we consider a modulated nonlinear CPEM wave (ω, k) , where the ponderomotive force induces slowly $(\partial_t \ll \omega)$ varying electrostatic oscillations. We still assume that the

motion is fast enough for the ions to be immobile. The evolution equation for the vector potential amplitude A in the Coulomb gauge can then be written as [36]

$$i\left(\partial_t + v_g\partial_z\right)A + \frac{1}{2}v_g'\partial_z^2 A - \left[\sum_{e,p}\frac{\omega\omega_p^2}{2(\omega+\omega_c)}\frac{v_g}{kc^2}\left(\frac{n_1}{n_0}\frac{\omega}{(\omega+\omega_c)} - \frac{e^2\omega^3|A|^2}{m_e^2c^2(\omega+\omega_c)^3}\right)\right]A = 0, \quad (5)$$

where the group velocity and dispersion are given above, and the indices e,p have been omitted for notational convenience. The electrostatic perturbations are described by the density fluctuation n_1, the longitudinal velocity v_z and the electrostatic potential ϕ. These variables satisfy the continuity equation

$$\frac{\partial n_1}{\partial t} + n_0\frac{\partial v_z}{\partial z} = 0, \quad (6)$$

and the momentum equation

$$m_e\frac{\partial v_z}{\partial t} = -qF - q\frac{\partial\phi}{\partial z} - \frac{T}{n_0}\frac{\partial n_1}{\partial z}, \quad (7)$$

where $q = -e$ for electrons and $q = e$ for positrons, together with

$$\frac{\partial^2\phi}{\partial z^2} = 4\pi e(n_{e1} - n_{p1}). \quad (8)$$

Here qF denotes the ponderomotive force, which can be written as [36]

$$qF = \frac{e^2\omega}{m_e(\omega+\omega_c)}\left(\partial_z + \frac{k\omega_c}{\omega(\omega+\omega_c)}\partial_t\right)|A|^2, \quad (9)$$

in a magnetized plasma. We assume that the temperature T is the same for electrons and positrons.

Next, we consider a high-frequency wave propagating with the group velocity, such that the driven low-frequency perturbations to first order are stationary functions of the comoving coordinate $z - v_g t$. Together with the assumption $\omega \ll \omega_c$, the ponderomotive force (9) then reduces to

$$\begin{aligned} qF &\simeq \frac{e^2\omega}{m_e\omega_c}\left(1 - \sqrt{1 + \frac{\Omega_i^2}{\omega\omega_{cp}}}\right)\partial_z|A|^2 \\ &\simeq -\frac{e^2\Omega_i^2}{2m_e\omega_c\omega_{cp}}\partial_z|A|^2, \end{aligned} \quad (10)$$

where we have used $\Omega_i^2/\omega\omega_c \ll 1$ in the last step. Next we expand Eq. (5) in the small parameter ω/ω_c to obtain

$$i\left(\partial_t + v_g\partial_z\right)A + \frac{1}{2}v_g'\partial_z^2 A - \left[\frac{\omega}{2\omega_{cp}^2}\sum_{e,p}\left[\omega_p^2\frac{n_1}{n_0}\left(1-\frac{2\omega}{\omega_c}\right) - \frac{e^2\omega^2\omega_p^2|A|^2}{m_e^2c^2\omega_c^2}\left(1-\frac{4\omega}{\omega_c}\right)\right]\right]A = 0. \tag{11}$$

As we can see, the correction term proportional to ω/ω_c must be kept above, since there will be an approximate cancellation of terms proportional to n_1/n_0 and $|A|^2$. Correction terms proportional to the small parameter $1 - v_g/c$ that have been omitted in Eq. (11) in general give an additional contribution to Eq. (11). However, unless we consider the regime where $\omega < \omega_p$ (in which case ion-motion must be included in the low-frequency dynamics), such terms are of higher order, and hence omitted. Next combining Eqs. (6)-(8) we can express the density perturbations in terms of the ponderomotive force and the potential. Substituting this expression in Eq. (5), using $\omega \ll \omega_c$ and $v_t^2 \equiv T/m_e \ll c^2$ to simplify the equation, we find

$$i\left(\partial_t + v_g\partial_z\right)A + \frac{1}{2}v_g'\partial_z^2 A + \frac{\omega}{2\omega_{cp}^2}\left[\frac{n_{0i}e^3}{m_e^2c^2}\phi A - \sum_{e,p}\frac{e^2\omega^2\omega_p^2}{m_e^2c^2\omega_c^2}\left(\frac{v_t^2}{c^2} - \frac{2\omega}{\omega_c}\right)|A|^2 A\right] = 0. \tag{12}$$

Next, combining Eqs. (6)-(8) to solve for the scalar potential, we have

$$\partial_z\left(\frac{\partial^2}{\partial t^2} - (v_e^2 + v_p^2)\frac{\partial^2}{\partial z^2} + \omega_{\text{ptot}}^2\right)\phi = -\frac{\Omega_i^2}{2m_e\omega_{cp}^2}\sum_{e,p} q\omega_p^2\frac{\partial|A|^2}{\partial z}, \tag{13}$$

where $\omega_{\text{ptot}}^2 = \omega_{pe}^2 + \omega_{pp}^2$. Integrating (13), dropping the term proportional to $(v_e^2 + v_p^2)/c^2$, and using the background charge balance we find

$$\left[\frac{\partial^2}{\partial t^2} + \omega_{\text{ptot}}^2\right]\phi = \frac{e\Omega_i^4|A|^2}{2m_e\omega_{cp}^2}. \tag{14}$$

The coupled equations (12) and (14) describe the nonlinear self-interactions of a high-frequency CPEM pulse combined with the generation of a low-frequency electrostatic wake field in a strongly magnetized electron-positron ion plasma. We note that there is no wake field generation in the absence of the ions.

IV. CONSERVATION LAWS

In a coordinate system moving with the group velocity and with suitable normalizations (i.e. $\xi = \omega_{\text{ptot}}(z - v_g t)/c$, etc.), the above system can be put into the generic form

$$\mathrm{i}\frac{\partial A}{\partial \tau} + \frac{\partial^2 A}{\partial \xi^2} = -A\left(\Phi - \alpha\,|A|^2\right) \tag{15}$$

$$\frac{\partial^2 \Phi}{\partial \xi^2} + \Phi = |A|^2\,, \tag{16}$$

where α is a constant that determines the relative importance of the self-nonlinearity, in our case given by

$$\alpha = \frac{8\pi\omega_{\text{ptot}}^2\omega^2}{\Omega_i^6}\left(\frac{\omega_{\text{ptot}}^2 v_t^2}{c^2} + \frac{2\Omega_i^2\omega}{\omega_{cp}}\right) \tag{17}$$

Various aspects of coupled system of this type have been studied by Refs. [37, 38, 39]. Following the presentation in [37] we will refer to the low-frequency field Φ as the wake-field. The equation system (15) and (16) can be derived from a variational principle. Introducing the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(A^*\frac{\partial A}{\partial \tau} - A\frac{\partial A^*}{\partial \tau}\right) - \left|\frac{\partial A}{\partial \xi}\right|^2 + \frac{1}{2}\alpha\,|A|^4 + \frac{1}{2}\left(\frac{\partial \Phi}{\partial \xi}\right)^2 - \frac{\Phi^2}{2} + |A|^2\Phi, \tag{18}$$

where the action functional is $\mathcal{A}(\Phi, A, A^*) = \int \mathcal{L} d\xi d\tau$, we obtain Eqs. (15) and (16) by varying Φ and A^* and minimizing the action as usual. The system posses the following three conservation laws:

$$\frac{d}{d\tau}\int_{-\infty}^{\infty} |A|^2\, d\xi = 0, \tag{19}$$

$$\frac{d}{d\tau}\int_{-\infty}^{\infty}\left(\left|\frac{\partial A}{\partial \xi}\right|^2 - \frac{\alpha}{2}\,|A|^4\,| - |A|^2\,\Phi\right) d\xi = 0, \tag{20}$$

$$\frac{d}{d\tau}\int_{\xi_-}^{\xi_+}\left(\frac{\partial A}{\partial \xi}A^* - \frac{\partial A^*}{\partial \xi}A\right) = W_\Phi|_{\xi_-}^{\xi_+}\,, \tag{21}$$

where $W_\Phi = (1/2)[\Phi^2 + (\partial\Phi/\partial\xi)^2]$. Within the framework of the WKB-approximation, $\int_{-\infty}^{\infty} |A|^2\, d\xi$ is proportional to the number of high-frequency quanta N, and thus Eq. (19) describes the conservation of N. Using a quantum mechanical analogue, the process of wake field generation can thus be viewed as a parametric process where a high-frequency quanta with frequency ω decays into a low frequency quanta Ω and another high-frequency quanta

with frequency $\omega - \Omega$, conserving the total number of high-frequency quanta. Furthermore, Eq. (20) is the conservation equation for the Hamiltonian. Although (21) does not look like a conservation law, it actually describes the conservation of energy. To see this, it is convenient to choose the positions $\xi_\pm$ before and after the pulse passage, respectively. For this choice we note that the left-hand side of (21) is proportional to $d/d\tau \int_{-\infty}^{\infty} \kappa |A_\kappa|^2 d\kappa / \int_{-\infty}^{\infty} |A_\kappa|^2 d\kappa = d\Delta k/d\tau = (1/v_g) d\Delta\omega/d\tau$. Furthermore, since N is conserved, $d\Delta\omega/d\tau$ is proportional to the high-frequency energy decay rate. Finally, the difference in W_Φ before and after the pulse passage is proportional to the energy transfer rate to the wake field, and thus we deduce that Eq. (21) implies energy conservation.

V. SUMMARY AND CONCLUSIONS

To summarize, we have considered the nonlinear interactions between magnetic field-aligned CPEM waves and electrostatic plasma oscillations (EPOs) in a pair-ion magnetoplasma. It is found that the EPOs are generated by the ponderomotive force of the CPEM waves only when the ions in a pair plasma are present. The present nonlinear wave-wave interactions provide the possibility of CPEM pulse localization and the generation of wakefields. Localized EM pulses, in association with wakefields, can be identified in observations from astrophysical settings as well as from next generation intense laser-solid density plasma experiments where pairs and ions appear simultaneously.

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